数理解析研究所講究録
第 1611 巻 2008 年 95-105

Weak and Strong convergence Theorems for Approximating common fixed Points of Three Nonexpansive Mappings

P. Glubudom and S. Suantai

Abstract : In this paper, a new three-step iterative scheme for three nonexpansives mappings is introduced and studied. Weak and strong convergence theorems of such iterations to a common fixed point of the nonexpansive mappings are established. The results obtained in this paper extend and improve the results due to [W. Takahashi, T. Tamura, Convergence theorems for a pair of nonexpansive mappings, J. Convex anal. 5(1995) 45-58], [K.K.Tan, H.K.Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal.Appl. 178(1993) 301-308], [H.F.Senter W.G.Dotson, Approximating fixed points of nonexpansive mappings, Proc.Amer.Math.Soc.44(1974) 375-380] and [G.Liu, D.Lei, S.Li, Approximating fixed points of nonexpansive mappings, Inernet.J.Math.Sci. 24(2000)173-177].

Keywords : Nonexpansive mapping, Common fixed point, Three-step iterative scheme, Opial property, weak and strong convergence.
2000 Mathematics Subject Classification : 47H09, 47J25.

1 Introduction

Let C be a nonempty convex subset of a real Banach space X, and let T_1, T_2 and $T_3 : C \to C$ be given mappings. Then for a given $x_1 \in C$, compute the sequence $\{x_n\}, \{y_n\}$ and $\{z_n\}$ by the iterative scheme

$$\begin{aligned} z_n &= a_n T_1 x_n + (1-a_n)x_n, \\ y_n &= b_n T_2 z_n + c_n T_1 x_n + (1-b_n-c_n)x_n, \\ x_{n+1} &= \alpha_n T_3 y_n + \beta_n T_2 z_n + \gamma_n T_1 x_n + (1-\alpha_n-\beta_n-\gamma_n)x_n, \end{aligned} \tag{1.1}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are appropriate sequences in $[0,1]$.

If $c_n = \beta_n = \gamma_n \equiv 0$ and $T_1 = T_2 = T_3$, then (1.1) reduces to the Noor iterations :

$$\begin{aligned} z_n &= a_n T_1 x_n + (1-a_n)x_n, \\ y_n &= b_n T_1 z_n + (1-b_n)x_n, \\ x_{n+1} &= \alpha_n T_1 y_n + (1-\alpha_n)x_n, \quad n \geq 1, \end{aligned} \tag{1.2}$$

where $\{a_n\}, \{b_n\}, \{\alpha_n\}$ are appropriate sequences in $[0,1]$.

If $a_n = b_n = \beta_n = \gamma_n \equiv 0$ and $T_1 = T_2 = T_3$, then (1.1) reduces to the usual Ishikawa iterative scheme

$$\begin{aligned} y_n &= c_n T_1 x_n + (1-c_n)x_n, \\ x_{n+1} &= \alpha_n T_1 y_n + (1-\alpha_n)x_n, \quad n \geq 1, \end{aligned}$$

where $\{c_n\}, \{\alpha_n\}$ are appropriate sequences in $[0,1]$.

If $T_1 = I$, the identity operator on C, and $\beta_n = 0$, then (1.1) reduces to the iterative scheme defined by Das and Debata [1] and Takahashi and Tomura [9]

$$\begin{aligned} y_n &= b_n T_2 x_n + (1-b_n)x_n, \\ x_{n+1} &= \alpha_n T_3 y_n + (1-\alpha_n)x_n, \quad n \geq 1, \end{aligned} \tag{1.3}$$

where $\{b_n\}, \{\alpha_n\}$ are sequences in $[0,1]$. Das and Debata [1] used the scheme (1.3) to approximate common fixed points of the maps when X is stricty convex. Takahashi and Tamura [9] prove weak convergence of the iterates $\{x_n\}$ defined by (1.3) in a uniformly convex Banach space X which satisfies the Opial property or whose norm is *Fre'chet* differentiable.

If $T_1 = I$, the identity operator on C, $\beta_n = 0$ and $T := T_2 = T_3$, then (1.1) reduces to the usual Ishikawa iterative scheme:

$$\begin{aligned} y_n &= b_n T x_n + (1-b_n)x_n, \\ x_{n+1} &= \alpha_n T y_n + (1-\alpha_n)x_n, \quad n \geq 1. \end{aligned}$$

If $T_1 = T_2 = I$ the identity operator on C and $T := T_3$, then (1.1) reduces to the usual Mann iterative scheme:

$$x_{n+1} = \alpha_n T x_n + (1-\alpha_n)x_n, \quad n \geq 1.$$

If $a_n = b_n = c_n \equiv 0$, then (1.1) reduces to the iterative scheme

$$\begin{aligned} x_1 &\in C, \\ x_{n+1} &= S_n x_n \quad n \geq 1, \end{aligned} \tag{1.4}$$

where $S_n = \alpha_n T_3 + \beta_n T_2 + \gamma_n T_1 + (1-\alpha_n-\beta_n-\gamma_n)I$.

If $\alpha_n = a, \beta_n = b$ and $\gamma_n = c$ for all $n \in N$, then (1.4) reduces to the iterative scheme defined by Liu, Lei and Li [3]

$$\begin{aligned} x_1 &\in C, \\ x_{n+1} &= S x_n \quad n \geq 1, \end{aligned} \tag{1.5}$$

where $S = aT_3 + bT_2 + cT_1 + (1-a-b-c)I$. Liu et al. [3] showed that $\{x_n\}$ defined by (1.5) converges to a common fixed point of T_1, T_2 and T_3 in Banach space, provided that $T_i (i = 1,2,3)$ satisfy condition A.

The purpose of this paper is to establish weak and strong convergence of the iterative scheme (1.1) to a common fixed point of three nonexpansive mappings in a uniformly convex Banach space.

Now, we recall the well-known concepts and results.

Let X be a normed space and C a nonempty subset of X. A mapping $T : C \to C$ is said to be *nonexpansive* on C if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

A Banach space X is said to satisfy *Opial's condition* if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ imply that

$$limsup_{n\to\infty}\|x_n - x\| < limsup_{n\to\infty}\|x_n - y\|.$$

In the sequel, the following lemmas are needed to prove our main results.

Lemma 1.1 *([5],*Lemma 4*) Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty), g(0) = 0$ such that*

$$\begin{aligned}\|\alpha x + \beta y + \gamma z + \lambda w\|^2 &\leq \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 + \lambda\|w\|^2 \\ &\quad -\frac{1}{3}\lambda(\alpha g(\|x - w\| + \beta g(\|y - w\| + \gamma g(\|z - q\|)),\end{aligned}$$

for all $x, y, z, w \in B_r$ and all $\alpha, \beta, \gamma, \lambda \in [0,1]$ with $\alpha + \beta + \gamma + \lambda = 1$.

Lemma 1.2 *([4],*Lemma 1.6*) Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X, and $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at 0, i.e., if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$, where $F(T)$ is the set of fixed point of T.*

Lemma 1.3 *([7],*Lemma 2.7*) Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $lim_{n\to\infty}\|x_n - u\|$ and $lim_{n\to\infty}\|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

2 Main results

In this section, we prove weak and strong convergence theorems of the iterative scheme (1.1) to a common fixed point of nonexpansive mappings T_1, T_2 and T_3. Let $F(t_i), i = 1,2,3$ denote the set of all fixed points of T_i, and let $F = \cap_{i=1}^{3} F(T_i)$. We first prove the following lammas.

Lemma 2.1 *Let X be a Banach space and C a nonempty closed and convex subset of X. Let T_1, T_2 and $T_3 : C \to C$ be nonexpansive self-maps and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be real sequences in $[0,1]$ such that $b_n + c_n$ and $\alpha_n + \beta_n + \gamma_n$ are in $[0,1]$ for all $n \geq 1$. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be sequences defined as in (1.1). If $F \neq \emptyset$ then $lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$.*

Proof. Let $p \in F$. Then

$$\begin{aligned} \|z_n - p\| &= \|a_nT_1x_n + (1-a_n)x_n - p\| \\ &\leq a_n\|T_1x_n - p\| + (1-a_n)\|x_n - p\| \\ &\leq a_n\|x_n - p\| + (1-a_n)\|x_n - p\| \\ &\leq \|x_n - p\| \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \|y_n - p\| &= \|b_nT_2z_n + c_nT_1x_n + (1-b_n-c_n)x_n - p\| \\ &\leq b_n\|T_2z_n - p\| + c_n\|T_1x_n - p\| + (1-b_n-c_n)\|x_n - p\| \\ &\leq b_n\|z_n - p\| + c_n\|x_n - p\| + (1-b_n-c_n)\|x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{2.2}$$

From (2.1) and (2.2), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_nT_3y_n + \beta_nT_2z_n + \gamma_nT_1x_n + (1-\alpha_n-\beta_n-\gamma_n)x_n - p\| \\ &\leq \alpha_n\|T_3y_n - p\| + \beta_n\|T_2z_n - p\| + \gamma_n\|T_1x_n - p\| \\ &\quad +(1-\alpha_n-\beta_n-\gamma_n)\|x_n - p\| \\ &\leq \alpha_n\|y_n - p\| + \beta_n\|z_n - p\| + \gamma_n\|x_n - p\| \\ &\quad +(1-\alpha_n-\beta_n-\gamma_n)\|x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{2.3}$$

Thus the sequence $\{\|x_n - p\|\}$ is bounded and decreasing which implies that $lim_{n\to\infty}\|x_n - p\|$ exists. ■

The next lemma is crucial for proving the main theorems.

Lemma 2.2 *Let X be a uniformly convex Banach space, and C a nonempty closed and convex subset of X. Let T_1, T_2 and $T_3 : C \to C$ be nonexpansive self-maps with $F \neq \emptyset$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}$ and $\{c_n\}$ be real sequences in $[0,1]$ such that $b_n + c_n$ and $\alpha_n + \beta_n + \gamma_n$ are in $[0,1]$ for all $n \geq 1$. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be sequences defined as in (1.1).*

(i) *If $0 < lim\ inf_{n\to\infty}\alpha_n$, $0 < lim\ inf_{n\to\infty}b_n$ and $0 < lim\ inf_{n\to\infty}a_n \leq lim\ sup_{n\to\infty}a_n < 1$, then $lim_{n\to\infty}\|T_1x_n - x_n\| = 0$.*

(ii) *If $0 < lim\ inf_{n\to\infty}c_n \leq lim\ sup_{n\to\infty}(b_n + c_n) < 1$ and $0 < lim\ inf_{n\to\infty}\alpha_n$, then $lim_{n\to\infty}\|T_1x_n - x_n\| = 0$.*

(iii) *If $0 < lim\ inf_{n\to\infty}a_n \leq lim\ sup_{n\to\infty}a_n < 1$ and $0 < lim\ inf_{n\to\infty}\beta_n$, then $lim_{n\to\infty}\|T_1x_n - x_n\| = 0$.*

(iv) *If $0 < lim\ inf_{n\to\infty}\gamma_n \leq lim\ sup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$, then $lim_{n\to\infty}\|T_1x_n - x_n\| = 0$.*

(v) *If* $0 < lim\ inf_{n\to\infty} b_n \le lim\ sup_{n\to\infty}(b_n + c_n) < 1$ *and* $0 < lim\ inf_{n\to\infty} a_n$, *then* $lim_{n\to\infty}\|T_2 z_n - x_n\| = 0$.

(vi) *If* $0 < lim\ inf_{n\to\infty} \beta_n \le lim\ sup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$, *then* $lim_{n\to\infty}\|T_2 z_n - x_n\| = 0$.

(vii) *If* $0 < lim\ inf_{n\to\infty} \alpha_n \le lim\ sup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$, *then* $lim_{n\to\infty}\|T_3 y_n - x_n\| = 0$.

Proof. Let $p \in F$. By Lemma 2.1, $sup_{n\ge1}\|x_n - p\|$ exists. Choose a number $r > 0$ and $r > sup_{n\ge1}\|x_n - p\|$, then by (2.1),(2.2),(2.3) we have that all sequences $\{z_n - p\}, \{y_n - p\}, \{x_n - p\}, \{T_1 x_n - p\}, \{T_2 z_n - p\}, \{T_3 y_n - p\}$ belong to B_r and by Lemma 1.1 there is a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$, such that

$$\begin{aligned}\|\alpha x + \beta y + \gamma z + \lambda w\|^2 &\le \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 + \lambda\|w\|^2 - \frac{1}{3}\alpha\lambda g(\|x - w\|)\\ &\quad -\frac{1}{3}\beta\lambda g(\|y - w\|) - \frac{1}{3}\gamma\lambda g(\|z - w\|) \qquad (2.4)\end{aligned}$$

for all $x, y, z, w \in B_r$ and all $\alpha, \beta, \gamma, \lambda \in [0,1]$ with $\alpha + \beta + \gamma + \lambda = 1$.

From (1.1) and (2.4) we have

$$\begin{aligned}\|z_n - p\|^2 &= \|a_n(T_1 x_n - p) + 0(0) + 0(0) + (1 - a_n)(x_n - p)\|^2\\ &\le a_n\|T_1 x_n - p\|^2 + (1 - a_n)\|x_n - p\|^2\\ &\quad -\frac{1}{3}a_n(1 - a_n)g(\|T_1 x_n - x_n\|)\\ &\le a_n\|x_n - p\|^2 + (1 - a_n)\|x_n - p\|^2\\ &\quad -\frac{1}{3}a_n(1 - a_n)g(\|T_1 x_n - x_n\|)\\ &= \|x_n - p\|^2 - \frac{1}{3}a_n(1 - a_n)g(\|T_1 x_n - x_n\|), \qquad (2.5)\end{aligned}$$

and

$$\begin{aligned}\|y_n - p\|^2 &= \|b_n(T_2 z_n - p) + c_n(T_1 x_n - p) + 0(0) + (1 - b_n - c_n)(x_n - p)\|^2\\ &\le b_n\|T_2 z_n - p\|^2 + c_n\|T_1 x_n - p\|^2 + (1 - b_n - c_n)\|x_n - p\|^2\\ &\quad -\frac{1}{3}(1 - b_n - c_n)[b_n g(\|T_2 z_n - x_n\|) + c_n g(\|T_1 x_n - x_n\|)]\\ &\le b_n\|z_n - p\|^2 + c_n\|x_n - p\|^2 + (1 - b_n - c_n)\|x_n - p\|^2\\ &\quad -\frac{1}{3}(1 - b_n - c_n)[b_n g(\|T_2 z_n - x_n\|) + c_n g(\|T_1 x_n - x_n\|)]\\ &\le b_n\|x_n - p\|^2 - \frac{1}{3}b_n a_n(1 - a_n)g(\|T_1 x_n - x_n\|)\\ &\quad +c_n\|x_n - p\|^2 + (1 - b_n - c_n)\|x_n - p\|^2\\ &\quad -\frac{1}{3}(1 - b_n - c_n)[b_n g(\|T_2 z_n - x_n\|) + c_n g(\|T_1 x_n - x_n\|)]\end{aligned}$$

$$
\begin{aligned}
&= \|x_n - p\|^2 - \frac{1}{3} b_n a_n (1 - a_n) g(\|T_1 x_n - x_n\|) \\
&\quad -\frac{1}{3}(1 - b_n - c_n)[b_n g(\|T_2 z_n - x_n\|) + c_n g(\|T_1 x_n - x_n\|)]. \qquad (2.6)
\end{aligned}
$$

By (1.1), (2.4), (2.5) and (2.6), we also have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n (T_3 y_n - p) + \beta_n (T_2 z_n - p) + \gamma_n (T_1 x_n - p) + \\
&\quad (1 - \alpha_n - \beta_n - \gamma_n)(x_n - p)\|^2 \\
&\le \alpha_n \|T_3 y_n - p\|^2 + \beta_n \|T_2 z_n - p\|^2 + \gamma_n \|T_1 x_n - p\|^2 \\
&\quad + (1 - \alpha_n - \beta_n - \gamma_n)\|(x_n - p)\|^2 \\
&\quad -\frac{1}{3}(1 - \alpha_n - \beta_n - \gamma_n)[\alpha_n g(\|T_3 y_n - x_n\|) + \beta_n g(\|T_2 z_n - x_n\|) \\
&\quad + \gamma_n g(\|T_1 x_n - x_n\|)] \\
&\le \alpha_n \|y_n - p\|^2 + \beta_n \|z_n - p\|^2 + \gamma_n \|x_n - p\|^2 \\
&\quad + (1 - \alpha_n - \beta_n - \gamma_n)\|(x_n - p)\|^2 \\
&\quad -\frac{1}{3}(1 - \alpha_n - \beta_n - \gamma_n)[\alpha_n g(\|T_3 y_n - x_n\|) + \beta_n g(\|T_2 z_n - x_n\|) \\
&\quad + \gamma_n g(\|T_1 x_n - x_n\|)] \\
&\le \alpha_n \|x_n - p\|^2 - \frac{1}{3}\alpha_n b_n a_n (1 - a_n) g(\|T_1 x_n - x_n\|) \\
&\quad -\frac{1}{3}\alpha_n (1 - b_n - c_n)[b_n g(\|T_2 z_n - x_n\|) + c_n g(\|T_1 x_n - x_n\|)] \\
&\quad + \beta_n \|x_n - p\|^2 - \frac{1}{3}\beta_n a_n (1 - a_n) g(\|T_1 x_n - x_n\|) + \gamma_n \|x_n - p\|^2 \\
&\quad + (1 - \alpha_n - \beta_n - \gamma_n)\|(x_n - p)\|^2 \\
&\quad -\frac{1}{3}(1 - \alpha_n - \beta_n - \gamma_n)[\alpha_n g(\|T_3 y_n - x_n\|) + \beta_n g(\|T_2 z_n - x_n\|) \\
&\quad + \gamma_n g(\|T_1 x_n - x_n\|)] \\
&= \|x_n - p\|^2 - \frac{1}{3}\alpha_n b_n a_n (1 - a_n) g(\|T_1 x_n - x_n\|) \\
&\quad -\frac{1}{3}\alpha_n (1 - b_n - c_n)[b_n g(\|T_2 z_n - x_n\|) + c_n g(\|T_1 x_n - x_n\|)] \\
&\quad -\frac{1}{3}\beta_n a_n (1 - a_n) g(\|T_1 x_n - x_n\|) \\
&\quad -\frac{1}{3}(1 - \alpha_n - \beta_n - \gamma_n)[\alpha_n g(\|T_3 y_n - x_n\|) + \beta_n g(\|T_2 z_n - x_n\|) \\
&\quad + \gamma_n g(\|T_1 x_n - x_n\|)]. \qquad (2.7)
\end{aligned}
$$

Thus

$$
\alpha_n b_n a_n (1 - a_n) g(\|T_1 x_n - x_n\|) \le 3[\|x_n - p\|^2 - \|x_{n+1} - p\|^2]. \qquad (2.8)
$$

(i) If $0 < lim\ inf_{n\to\infty}\alpha_n$, $0 < lim\ inf_{n\to\infty}b_n$ and $0 < lim\ inf_{n\to\infty}a_n \leq lim\ sup_{n\to\infty}a_n < 1$, then there exist positive integer n_0 and reals $\eta_1, \eta_2, \eta_3, \eta_4 \in (0,1)$ such that $0 < \eta_1 \leq \alpha_n$, $0 < \eta_2 \leq b_n$, $0 < \eta_3 \leq a_n < \eta_4 < 1$ for all $n \geq n_0$. It follows from (2.8) that

$$\eta_1\eta_2\eta_3(1-\eta_4)g(\|T_1x_n - x_n\|) \leq \ 3[\|x_n - p\|^2 - \|x_{n+1} - p\|^2] \quad for\ all\ n \geq n_0.$$

This implies by Lemma 2.1 that $\lim_{n\to\infty} g(\|T_1x_n - x_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\to\infty} \|T_1x_n - x_n\| = 0$.

By using (2.7) and Lemma 2.1 with the same method as in (i), then (ii)-(vii) are directly obtained, respectively. ■

Lemma 2.3 *Let X be a uniformly convex Banach space, and C a nonempty closed and convex subset of X. Let T_1, T_2 and $T_3 : C \to C$ be nonexpansive self-maps of C with $F \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}$ and $\{c_n\}$ be real sequences in $[0,1]$ such that $b_n + c_n$ and $\alpha_n + \beta_n + \gamma_n$ are in $[0,1]$ for all $n \geq 1$. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences defined by the iterative scheme (1.1) if*

(i) $0 < lim\ inf_{n\to\infty}\alpha_n \leq lim\ sup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$,
$0 < lim\ inf_{n\to\infty}b_n \leq lim\ sup_{n\to\infty}(b_n + c_n) < 1$ *and*
$0 < lim\ inf_{n\to\infty}a_n \leq lim\ sup_{n\to\infty}a_n < 1$, *or*

(ii) $0 < lim\ inf_{n\to\infty}\alpha_n \leq lim\ sup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$,
$0 < min\{lim\ inf_{n\to\infty}b_n, lim\ inf_{n\to\infty}c_n\} \leq lim\ sup_{n\to\infty}(b_n + c_n) < 1$, *or*

(iii) $0 < min\{lim\ inf_{n\to\infty}\alpha_n, lim\ inf_{n\to\infty}\beta_n\} \leq lim\ sup_{n\to\infty}(\alpha_n+\beta_n+\gamma_n) < 1$,
$0 < lim\ inf_{n\to\infty}b_n \leq lim\ sup_{n\to\infty}(b_n + c_n) < 1$ *and*
$0 < lim\ inf_{n\to\infty}a_n \leq lim\ sup_{n\to\infty}a_n < 1$, *or*

(iv) $0 < min\{lim\ inf_{n\to\infty}\alpha_n, lim\ inf_{n\to\infty}\gamma_n\} \leq lim\ sup_{n\to\infty}(\alpha_n+\beta_n+\gamma_n) < 1$,
$0 < lim\ inf_{n\to\infty}b_n \leq lim\ sup_{n\to\infty}(b_n + c_n) < 1$ *or*

(v) $0 < min\{lim\ inf_{n\to\infty}\alpha_n, lim\ inf_{n\to\infty}\beta_n\} \leq lim\ sup_{n\to\infty}(\alpha_n+\beta_n+\gamma_n) < 1$,
$0 < lim\ inf_{n\to\infty}a_n \leq lim\ sup_{n\to\infty}a_n < 1$, *and*
$0 < lim\ inf_{n\to\infty}b_n$, *or*

(vi) $0 < min\{lim\ inf_{n\to\infty}\alpha_n, lim\ inf_{n\to\infty}\beta_n\} \leq lim\ sup_{n\to\infty}(\alpha_n+\beta_n+\gamma_n) < 1$,
$0 < lim\ inf_{n\to\infty}c_n \leq lim\ sup_{n\to\infty}(b_n + c_n) < 1$, *or*

(vii) $0 < min\{lim\ inf_{n\to\infty}\alpha_n, lim\ inf_{n\to\infty}\beta_n\} \leq lim\ sup_{n\to\infty}(\alpha_n+\beta_n+\gamma_n) < 1$,
$0 < lim\ inf_{n\to\infty}a_n \leq lim\ sup_{n\to\infty}a_n < 1$, *or*

(viii) $0 < min\{lim\ inf_{n\to\infty}\alpha_n, lim\ inf_{n\to\infty}\beta_n, lim\ inf_{n\to\infty}\gamma_n\}$
$\leq lim\ sup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1,$

then $\lim_{n\to\infty} \|T_1x_n - x_n\| = \lim_{n\to\infty} \|T_2x_n - x_n\| = \lim_{n\to\infty} \|T_3x_n - x_n\| = 0.$

Proof. (i) By Lemma 2.2, we have

$$\lim_{n\to\infty} \|T_1x_n - x_n\| = 0, \lim_{n\to\infty} \|T_2z_n - x_n\| = 0, \lim_{n\to\infty} \|T_3y_n - x_n\| = 0.$$

It follows that

$$\begin{aligned} \|T_2x_n - x_n\| &\leq \|T_2x_n - T_2z_n\| + \|T_2z_n - x_n\| \\ &\leq \|z_n - x_n\| + \|T_2z_n - x_n\| \\ &= \|a_nT_1x_n + (1 - a_n)x_n - x_n\| + \|T_2z_n - x_n\| \\ &\leq a_n\|T_1x_n - x_n\| + \|T_2z_n - x_n\| \\ &\leq \|T_1x_n - x_n\| + \|T_2z_n - x_n\| \to 0 \qquad \textit{as } n \to \infty, \textit{and} \end{aligned}$$

$$\begin{aligned} \|T_3x_n - x_n\| &\leq \|T_3x_n - T_3y_n\| + \|T_3y_n - x_n\| \\ &\leq \|x_n - y_n\| + \|T_3y_n - x_n\| \\ &= \|b_nT_2z_n + c_nT_1x_n + (1 - b_n - c_n)x_n - x_n\| + \|T_3y_n - x_n\| \\ &\leq b_n\|T_2z_n - x_n\| + c_n\|T_1x_n - x_n\| + \|T_3y_n - x_n\| \\ &\leq \|T_2z_n - x_n\| + \|T_1x_n - x_n\| + \|T_3y_n - x_n\| \to 0 \quad \textit{as } n \to \infty. \end{aligned}$$

By using the same proof as in (i), (ii)- (viii) are obtained. ■

Theorem 2.4 *Let X be a uniformly convex Banach space, and C a nonempty closed and convex subset of X. Let T_1, T_2 and $T_3 : C \to C$ be nonexpansive self-maps of C with $F \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}$ and $\{c_n\}$ be real sequences in $[0, 1]$ such that $b_n + c_n$ and $\alpha_n + \beta_n + \gamma_n$ are in $[0, 1]$ for all $n \geq 1$. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences defined by the iterative scheme (1.1) if*

(i) $0 < lim\ inf_{n\to\infty}\alpha_n \leq lim\ sup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1,$
$0 < lim\ inf_{n\to\infty}b_n \leq lim\ inf_{n\to\infty}(b_n + c_n) < 1$ *and*
$0 < lim\ inf_{n\to\infty}a_n \leq lim\ sup_{n\to\infty}a_n < 1,$ *or*

(ii) $0 < lim\ inf_{n\to\infty}\alpha_n \leq lim\ sup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1,$
$0 < min\{lim\ inf_{n\to\infty}b_n, lim\ inf_{n\to\infty}c_n\} \leq lim\ inf_{n\to\infty}(b_n + c_n) < 1,$ *or*

(iii) $0 < min\{lim\ inf_{n\to\infty}\alpha_n, lim\ inf_{n\to\infty}\beta_n\} \leq lim\ sup_{n\to\infty}(\alpha_n+\beta_n+\gamma_n) < 1,$
$0 < lim\ inf_{n\to\infty}b_n \leq lim\ inf_{n\to\infty}(b_n + c_n) < 1$ *and*
$0 < lim\ inf_{n\to\infty}a_n \leq lim\ sup_{n\to\infty}a_n < 1,$ *or*

(iv) $0 < min\{lim\ inf_{n\to\infty}\alpha_n, lim\ inf_{n\to\infty}\gamma_n\} \leq lim\ sup_{n\to\infty}(\alpha_n+\beta_n+\gamma_n) < 1$, $0 < lim\ inf_{n\to\infty}b_n \leq lim\ inf_{n\to\infty}(b_n + c_n) < 1$ *or*

(v) $0 < min\{lim\ inf_{n\to\infty}\alpha_n, lim\ inf_{n\to\infty}\beta_n\} \leq lim\ sup_{n\to\infty}(\alpha_n+\beta_n+\gamma_n) < 1$, $0 < lim\ inf_{n\to\infty}a_n \leq lim\ sup_{n\to\infty}a_n < 1$, *and* $0 < lim\ inf_{n\to\infty}b_n$, *or*

(vi) $0 < min\{lim\ inf_{n\to\infty}\alpha_n, lim\ inf_{n\to\infty}\beta_n\} \leq lim\ sup_{n\to\infty}(\alpha_n+\beta_n+\gamma_n) < 1$, $0 < inf_{n\to\infty}c_n \leq lim\ inf_{n\to\infty}(b_n + c_n) < 1$, *or*

(vii) $0 < min\{lim\ inf_{n\to\infty}\alpha_n, lim\ inf_{n\to\infty}\beta_n\} \leq lim\ sup_{n\to\infty}(\alpha_n+\beta_n+\gamma_n) < 1$, $0 < lim\ inf_{n\to\infty}a_n \leq lim\ sup_{n\to\infty}a_n < 1$, *or*

(viii) $0 < min\{lim\ inf_{n\to\infty}\alpha_n, lim\ inf_{n\to\infty}\beta_n, lim\ inf_{n\to\infty}\gamma_n\}$ $\leq lim\ sup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$,

and one of T_1, T_2 and T_3 is completely continuous, then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a common fixed point of T_1, T_2 and T_3.

Proof. (i) By lemma 2.3, we have

$$\lim_{n\to\infty} \|T_1x_n - x_n\| = \lim_{n\to\infty} \|T_2x_n - x_n\| = \lim_{n\to\infty} \|T_3x_n - x_n\| = 0. \tag{2.9}$$

Suppose without loss of generality that T_1 is completely continuous. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{T_1x_{n_k}\}$ converges. Therefore from (2.9), $\{x_{n_k}\}$ converges. Let $\lim_{n\to\infty} x_{n_k} = q$. By continuity of T_1 and (2.9) we have that $T_1q = q$, so q is a fixed point of T_1. Since T_2, T_3 are continuous and $\lim_{n\to\infty} \|T_2x_n - x_n\| = \lim_{n\to\infty} \|T_3x_n - x_n\| = 0$, we obtain that $q \in F(T_2), q \in F(T_3)$, so $q \in F$. By Lemma 2.1, $\lim_{n\to\infty} \|x_n - q\|$ exists. But $\lim_{n\to\infty} x_{n_k} = q$, so $\lim_{n\to\infty} x_n = q$.

$$\begin{aligned} \textit{Since} \quad \|y_n - x_n\| &\leq b_n\|T_2z_n - x_n\| + c_n\|T_1x_n - x_n\| \to 0 \quad \textit{and} \\ \|z_n - x_n\| &= a_n\|T_1x_n - x_n\| \to 0 \qquad \textit{as } n \to \infty, \end{aligned}$$

it follows that $\lim_{n\to\infty} y_n = q$ and $\lim_{n\to\infty} z_n = q$

The proof of (ii)-(viii) is similar to that of (i). ■

For $c_n = \beta_n = \gamma_n = 0$ for all $n \in N$, the following result are obtained directly from Theorem 2.4.

Corollary 2.5 *Let X be a uniformly convex Banach space, and C a nonempty closed and convex subset of X. Let T_1, T_2 and $T_3 : C \to C$ be nonexpansive self-maps of C with $F \neq \emptyset$. Let $\{a_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}$, and $\{z_n\}$ be the sequences defined by the iterative scheme (1.2).*

$$\begin{aligned} If \quad 0 &< lim\ inf_{n\to\infty} a_n \leq lim\ sup_{n\to\infty} a_n < 1, \\ 0 &< lim\ inf_{n\to\infty} b_n \leq lim\ sup_{n\to\infty} b_n < 1, \\ 0 &< lim\ inf_{n\to\infty} \alpha_n \leq lim\ sup_{n\to\infty} \alpha_n < 1 \ \ and \end{aligned}$$

one of T_1, T_2 and T_3 is completely continuous, then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a common fixed point of T_1, T_2 and T_3.

In the next result, we prove weak convergence for the iterative scheme (1.1) for three nonexpansive mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 2.6 *Let X be a uniformly convex Banach space which satisfies Opial's condition, and C a nonempty closed and convex subset of X. Let T_1, T_2 and $T_3 : C \to C$ be nonexpansive self-maps of C with $F \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}$ and $\{c_n\}$ be real sequences in $[0,1]$ such that $b_n + c_n$ and $\alpha_n + \beta_n + \gamma_n$ are in $[0,1]$ for all $n \geq 1$. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be sequences defined by the iterative scheme (1.1)*

(i) *If* $0 < lim\ inf_{n\to\infty} \alpha_n \leq lim\ sup_{n\to\infty} (\alpha_n + \beta_n + \gamma_n) < 1,$
$0 < lim\ inf_{n\to\infty} \beta_n \leq lim\ sup_{n\to\infty} (\alpha_n + \beta_n + \gamma_n) < 1,$ *and*
$0 < lim\ inf_{n\to\infty} \gamma_n \leq lim\ sup_{n\to\infty} (\alpha_n + \beta_n + \gamma_n) < 1,$
then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge weakly to a common fixed point of T_1, T_2 and T_3.

(ii) *If* $0 < lim\ inf_{n\to\infty} a_n \leq lim\ sup_{n\to\infty} a_n < 1,$
$0 < lim\ inf_{n\to\infty} b_n \leq lim\ sup_{n\to\infty} (b_n + c_n) < 1,$ *and*
$0 < lim\ inf_{n\to\infty} \alpha_n \leq lim\ sup_{n\to\infty} (\alpha_n + \beta_n + \gamma_n) < 1,$
then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge weakly to a common fixed point of T_1, T_2 and T_3.

Proof. (i) If follows from Lemma 2.3 that

$$\lim_{n\to\infty} \|T_1 x_n - x_n\| = \lim_{n\to\infty} \|T_2 x_n - x_n\| = \lim_{n\to\infty} \|T_3 x_n - x_n\| = 0.$$

Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to u$ weakly as $n \to \infty$, without loss of generality. By Lemma 1.4, we have $u \in F$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. From Lemma 1.2, $u, v \in F$. By Lemma 2.1, $lim_{n\to\infty}\|x_n - u\|$ and $lim_{n\to\infty}\|x_n - v\|$ exist. It follows from Lemma 1.3 that $u = v$. Therefor $\{x_n\}$ converge weakly to a common fixed point of T_1, T_2 and T_3.

(ii) The proof of (ii) is similar to that of (i). ■

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